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On configuration of Limit Cycles in certain planner vector fields

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Let X_λ be a one parameter family of vector field on the plane satisfying $Det(X_{\lambda_1}, X_{\lambda_2}) > 0$ for $\lambda_1 > \lambda_2$.
This means every solution of X_{λ_1} is transverse to solutions of X_{λ_2}. We call X_λ a family of rotated vector fields (Note that such family can be defined on any symplictic manifold,and each X_λ is transvers to isotropic or lagrangian submanifold invariant under a X_{λ_0} thus it would be interesting to equip a symlictic manifold to new volume symlictic form, in order to facilitate in working with a family which is not "rotated family" with respect to usual symplictic form). This phenomenon have been presented by Duff [1]. In this note, we prove three observation using the properties of rotated families (In third observation, however, we do not have a rotated family, but the argument is similar to the methods in rotated vector fields).
Proposition i. The quadratic system

$$\begin{cases} \dot{x} = y + ax^2 + by^2 + cx \\ \dot{y} = -x + dx^2 + fxy \end{cases} \tag{1}$$

can not have two limit cycles with disjoint interiors.
Proposition ii. The Lienard system

$$\begin{cases} \dot{x} = y + ax^5 + bx^3 + cx \\ \dot{y} = -x \end{cases} \tag{2}$$

has a semistable limit cycle if and only if $bc < 0$ and $a = \phi(b, c)$, where ϕ is a unique analytic function.

Proposition iii. Lienard system

$$\begin{cases} \dot{x} = y + (x^4 - 2x^2) \\ \dot{y} = \varepsilon(a - x) \end{cases} \tag{3}$$

has at least one limit cycle if and only if $0 < |a| < 1$.

Remark. Proposition 1 could be in particular due to question posed in [8] about coexistence of two limit cycles with disjoint interior in quadratic system. Proposition 2 is actually giving a partial answer to a question about multiplicity of limit cycle in Rychkov-Lienard system (see[9-page 261], and [6]). Proposition 3 would try suggesting a counterexample of a system

$$\begin{cases} \dot{x} = y - (ax^4 + bx^3 + cx^2 + dx) \\ \dot{y} = -x \end{cases}$$

with at least two limit cycles. See conjecture in [3] about system (3), it seems that no duck limit cycles could be existed (Due to intuitions from canard solutions described in [2]). From other hand Proposition 3 assert that we have at least one limit cycle. This shows that perhaps for ε and a small, the limit cycles bifurcate from infinity, however the minimum values of y-coordinates of such limit cycle(s) can not be less than -1, using Remark 3 in [7]. Thus it would be interesting investigation of the number of limit cycles of (3) or adding a term εx^3 to first line of (3). I thank Professor Roussarie that he explained about canard solutions and suggested the latest system as a possibly candidate for counterexample to Pugh's conjecture.

Proof of Proposition 1. This is proved in three steps

i. If a limit cycle surround the origin then $cd(2a + f) > 0$,

ii. If a limit cycle does not intersect the line x=0 and has positive (negative) orintion then $cd(2a + f) < 0, (> 0)$,

iii. If $cd(2a + f) = 0$ then two limit cycles can not coexist.

Assume that all 3 statements in above are proved,let γ_1 ,γ_2 are two limit cycles with disjoint interiors, by **ii** and **iii** at least one of the γ_1 and γ_2 must intersect the y-axis and we may assume that the origin lies in γ_1 (for if not we translate the singularity inside of γ_1 to origin. From **i** and **ii** we obtain that γ_2 must also intersect the line $x = 0$. Therefore Both γ's do not intersect the line $-1+dx+fy = 0$ because any closed orbit of a quadratic system can surround only one singularity[10]. Now we add $-cx$ to first equation of (1) and we obtain a limit cycle for (1) when $c = 0$, while is impossible, see [9-page 315].

Proof of step iii. when $c = 0$, (1) does not have a limit cycle because of the reason mentioned in above two line, if $2a + f = 0$, divergence of (1) is constant thus there is non limit cycle and if $d = 0$, we have at most one limit cycle, see [9], in which the origin does not lie because for $c = 0$ and $d = 0$ the origin is a center: (note that in a rotated vector field family, if we have a center for a parameter λ_0 we could not have limit cycle for other values of λ.

Proof of step i. If the origin lies inside a limit cycle then $cd(2a+f)$ is not 0 and if it is negative we add $-cx$ to first equation and a contradiction is follows.

Proof of step ii. Note that if a limit cycle does not intersect the y-axis,then x values of its point has the same sign as the $\frac{-c}{2a+f}$ and by computation of $\int_\gamma(-1+dx+fy)dy$ we find that it has the same sign as d (for positive orient of parameterizations of limit cycle γ, then $cd(2a + f)$ is negative. Similar consideration hold for negative orient and the proof is completed.

Proof Of Proposition ii. It is proved in [6] that system (2) has at most two limit cycles. In fact this result is true counting multiplicity: Let $P(y)$ be the poincare map defined on positive y-axis. Then $P'(y) = \frac{y}{P(y)}e^{h(y)}$ where $h(y) = \int_0^{T(y)}$ divergence of(2), $T(y)$ is the time of first return.

Assume that $P(y) = y_0$ and $P'(y_0) = 1$, the computation in [6], actually shows that $h'(y_0) \neq 0$ so $P''(y_0) \neq 0$ then (2) has at most two limit cycle counting multiplicity. Now We present a global bifurcation diagram of semi-stable limit cycle for (2). If $bc > 0$ then by lienard theorem [5], there is no semistable limit cycle. Assume that $bc < 0$. For $a = 0$, system (2) has a unique hyperbolic limit cycle. We can assume $c < 0$ and $b > 0$, if $a < 0$ and $|a| \lll 1$, then another limit cycle would born at infinity. If for some $a_0 < 0$, two limit cycles exist, then the same holds for $a_0 < a < 0$, because if γ_1 and γ_2 would be two limit cycles for $(2)_{a_0}$, then both of γ_1 and γ_2 are closed curve without contact for $(2)_a$ for all $a_0 < a < 0$. Now Compare the direction of $(2)_a$ on γ_1 and γ_2 with stability of origin and infinity. On the other hand for fixed $c < 0$, $b > 0$, if $|a|$ is sufficiently large ($a < 0$), then the derivative of energy does not change sign. Therefore there exist a unique $a_0 = \phi(b, c)$ such that (2) has a semistable limit cycle. a_0 is unique because from any semiustable limit cycle, two limit cycles could be created. Now all conditions of Theorem 2 in [4] satisfy and proposition 2 is proved.

Proof of proposition iii. For $|a| \geq 1$ there is no limit cycle using proposition in [3], after change of coordinate $x := x + a, y := y + a^4 - 2a^2$.

For $a = 0$ the system (3) has a center whose region of closed orbits is bounded by a unique orbit γ asymptotic to the graph of $y = x^4 - 2x^2$ and γ is below

this graph, thus γ is a curve without contact for $(3)_a$ and the singularity is attractive. Thus Poincare Bendixon theorem convert to existence of at least one limit cycles.

Remark. It Is obvious that a multiple limit cycle (with arbitrary finite large multiplicity) can produce at most two limit cycles with one parameter perturbation in a rotated family. How much this results remain valid in the case of infinite multiplicity? See [5-page 387].

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